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1990 J. Phys. A: Math. Gen. 23 3991

(http://iopscience.iop.org/0305-4470/23/17/028)

J. Phys. A: Math. Gen. 23 (1990) 3991-3996. Printed in the UK

Causality and the scalar field energy

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Received 31 January 1990, in final form 1 June 1990

Abstract. The evolution of the scalar field energy density is considered. It is shown that the total energy outside an expanding light cone does not increase. This result is extended to a general density in n dimensions, when certain conditions are satisfied. The apparent conflict with previous proofs of acausal behaviour for relativistic densities is discussed.

1. Introduction

Acausal behaviour in relativistic quantum mechanics has its roots in the problem of localisation. Newton and Wigner [1] showed how to obtain a localised state, only requiring that if this state is translated then it is orthogonal to the original state, that is their scalar product is zero. Hegerfeldt [2] and others [3] then proved that for any $t>0$, the original and translated states cease to be orthogonal however far the translation. Hence there appears to be instantaneous spreading of the Newton-Wigner probability density. Any superluminal spreading is acausal, as motion outside the light cone into the spacelike separated region may appear to another observer as proceeding backwards in time (see the article by Feynman in [4]). More recently Rosenstein and Usher [5] have demonstrated superluminal spreading for a particular closed-form solution of the positive-energy Klein-Gordon equation.

The underlying reason for this behaviour is in the nature of the time evolution operator for a positive-energy relativistic particle. In momentum space this operator has the simple form:

$$\exp(-\mathrm{i}\omega t)=\exp[-\mathrm{i}t(m^2+p^2)^{1/2}].$$

However, in configuration space the ω operator transforms to the operator

$$\hat{\omega}=F^{-1}\omega F=(m^2-\nabla^2)^{1/2}$$

in our units (where F is the Fourier transform operator). As discussed below this operator is non-local and so allows instantaneous spreading. This is also true of the evolution operator $\exp(-\mathrm{i}\hat{\omega}t)$ (see, for example, Hegerfeldt [2]).

Our approach is to consider the energy density of the (complex) scalar field, namely

$$\tfrac{1}{2}(|(\partial/\partial t)\psi|^2+|(\partial/\partial x)\psi|^2+m^2|\psi|^2)$$

(throughout we use units where $c=1$), where ψ evolves according to the Klein-Gordon equation. We shall show that the total energy outside the light cone cannot increase, and relate this to causality.

2. Evolution of the energy density in 1+1 dimensions

The scalar field energy is taken as $H=\frac{1}{2}(|\partial_0\psi|^2+|\partial_x\psi|^2+m^2|\psi|^2)$ which is obviously positive definite (here ∂_μ stands for $\partial/\partial x^\mu$), and the momentum is

$$P=-\tfrac{1}{2}(\partial_0\psi^*\partial_x\psi+\partial_x\psi^*\partial_0\psi).$$

We define $H_{\text{out}}(t)$ to be the integrated energy outside the light cone at time t. Suppose the light cone occupies the interval $|x|<d$ at $t=0$. We consider the energy density for $x>0$ for simplicity. Then

$$H_{\text{out}}(0)=\int_d^\infty H(0,x)\,\mathrm{d}x=\int_d^\infty \tfrac{1}{2}(|\partial_0\psi|^2+|\partial_x\psi|^2+m^2|\psi|^2)\,\mathrm{d}x \tag{2.1a}$$

and

$$H_{\text{out}}(t)=\int_{d+t}^\infty H(t,x)\,\mathrm{d}x=\int_{d+t}^\infty \tfrac{1}{2}(|\partial_0\psi|^2+|\partial_x\psi|^2+m^2|\psi|^2)\,\mathrm{d}x \tag{2.1b}$$

so the time change in $H_{\text{out}}(t)$ is

$$\partial_0 H_{\text{out}}(t)=\partial_0\int_{d+t}^\infty H(t,x)\,\mathrm{d}x=-H(t,d+t)+\int_{d+t}^\infty \partial_0 H(t,x)\,\mathrm{d}x. \tag{2.2}$$

To evaluate $\partial_0 H$ in the second term we note that H and P satisfy the continuity equation:

$$\partial_0 H+\partial_x P=0 \tag{2.3}$$

as is well known and may easily be verified. Thus

$$\begin{aligned}\partial_0 H_{\text{out}}(t)&=-H(t,d+t)-\int_{d+t}^\infty \partial_x P(t,x)\,\mathrm{d}x\\ &=-H(t,d+t)+P(t,d+t)\end{aligned} \tag{2.4}$$

assuming that P at infinity is zero.

Thus the rate of change of the integrated energy outside a specified light cone is equal to the momentum density minus the energy density at the edge of the light cone. Only the energy density and momentum density have to be calculated at a point to see how the total energy outside the light cone from that point is changing with time. Equation (2.4) tells us that if $H>|P|$ at the edge of the light cone, then the total energy outside the light cone is decreasing, as we would expect for causal behaviour. Conversely $P>H$ would imply that the energy outside the light cone is increasing, in contradiction to causality.

Note. Equation (2.4) is relevant for any density provided only that (i) a continuity equation is satisfied, and (ii) the current disappears at infinity. In general, these conditions can always be satisfied, although unless the current is known directly there is little gain in using (2.4). We later extend the relation (2.4) to n dimensions.

We now prove that the energy density does indeed exceed the momentum density at any point:

$$\begin{aligned}4(H^2-P^2)&=(|\partial_0\psi|^2+|\partial_x\psi|^2+m^2|\psi|^2)^2-(\partial_0\psi^*\partial_x\psi+\partial_x\psi^*\partial_0\psi)^2\\ &=(\partial_0\psi^*\partial_0\psi)^2+(\partial_x\psi^*\partial_x\psi)^2+2(\partial_0\psi^*\partial_0\psi\partial_x\psi^*\partial_x\psi)\\ &\quad+m^2|\psi|^2(2|\partial_0\psi|^2+2|\partial_x\psi|^2+m^2|\psi|^2)\\ &\quad-(\partial_0\psi^*)^2(\partial_x\psi)^2-(\partial_x\psi^*)^2(\partial_0\psi)^2-2(\partial_0\psi^*\partial_0\psi\partial_x\psi^*\partial_x\psi)\\ &=(\partial_0\psi^*)^2(\partial_0\psi)^2+(\partial_x\psi^*)^2(\partial_x\psi)^2-(\partial_0\psi^*)^2(\partial_x\psi)^2-(\partial_x\psi^*)^2(\partial_0\psi)^2\\ &\quad+m^2|\psi|^2(2|\partial_0\psi|^2+2|\partial_x\psi|^2+m^2|\psi|^2).\end{aligned}$$

Now the first line of the above expression is non-negative by Schwarz's inequality and the second line is positive definite, so $H^2 - P^2 \geqslant 0$ hence $H \geqslant |P|$. We can see from (2.4) that energy cannot increase outside the light cone. This is our result in $1+1$ dimensions.

We shall now turn to the $(n+1)$-dimensional case: here we show that the total energy inside the light cone cannot decrease—which by energy conservation is equivalent to proving that the total energy outside the light cone cannot increase.

3. Extension to $n+1$ dimensions

We first derive a result for a general density ρ in n space dimensions, before returning to the energy density.

Let $V(t)$ be a region of space, depending on time t, defined as follows: (i) $V(t)$ is a finite region, (ii) if $t+\delta t > t \geqslant t_0$, then $V(t+\delta t)$ is obtained from $V(t)$ by allowing each point of the surface bounding $V(t)$ to travel along the outward normal a distance δt. Let ρ be a positive-definite density, and $\boldsymbol{J}$ an associated current satisfying the continuity equation $\partial_0 \rho + \nabla \cdot \boldsymbol{J} = 0$, equivalently (using the repeated suffix notation)

$$\partial_0 \rho + \partial_a J^a = 0. \tag{3.1}$$

Define $M(t)$ as the density integrated over the region $V(t)$; then

$$\delta M(t) = \int_{V(t+\delta t)} \rho(\boldsymbol{x}, t+\delta t)\, \mathrm{d}V - \int_{V(t)} \rho(\boldsymbol{x}, t)\, \mathrm{d}V$$

and therefore

$$\delta M(t) = \left(\int_{V(t+\delta t)} \rho(\boldsymbol{x}, t+\delta t)\, \mathrm{d}V - \int_{V(t)} \rho(\boldsymbol{x}, t+\delta t)\, \mathrm{d}V \right) + \left(\int_{V(t)} \rho(\boldsymbol{x}, t+\delta t)\, \mathrm{d}V - \int_{V(t)} \rho(\boldsymbol{x}, t)\, \mathrm{d}V \right).$$

If δt is small the first term may be approximated by

$$\int_{S(t)} \rho(\boldsymbol{x}, t+\delta t)\, \mathrm{d}S\, \delta t \sim \int_{S(t)} \rho(\boldsymbol{x}, t)\, \mathrm{d}S\, \delta t$$

where $S(t)$ is the surface of $V(t)$, therefore

$$\frac{\delta M(t)}{\delta t} = \int_{S(t)} \rho(\boldsymbol{x}, t)\, \mathrm{d}S + \int_{V(t)} \frac{\rho(\boldsymbol{x}, t+\delta t) - \rho(\boldsymbol{x}, t)}{\delta t}\, \mathrm{d}V.$$

Letting $\delta t \to 0$

$$\dot{M} = \int_{S(t)} \rho(\boldsymbol{x}, t)\, \mathrm{d}S + \int_{V(t)} \frac{\partial \rho(\boldsymbol{x}, t)}{\partial t}\, \mathrm{d}V.$$

Now using (3.1) we obtain

$$\dot{M} = \int_{S(t)} \rho(\boldsymbol{x}, t)\, \mathrm{d}S - \int_{V(t)} \partial_a J^a\, \mathrm{d}V.$$

We can use the divergence theorem in n dimensions to evaluate the second term above as a surface integral; this gives

$$\dot{M} = \int_{S(t)} [\rho(\boldsymbol{x}, t) - J_n(\boldsymbol{x}, t)]\, dS \tag{3.2}$$

where J_n is the normal component of $\boldsymbol{J}$ out of $V(t)$. Now suppose:

(i) $\rho(\boldsymbol{x}, t) \geqslant 0 \quad \forall \quad \boldsymbol{x}, t$

(ii) $\rho(\boldsymbol{x}, t) \geqslant |J(\boldsymbol{x}, t)| \quad \forall \quad \boldsymbol{x}, t$

then (3.2) shows that $\dot{M} \geqslant 0\ \forall t$. This is our result in n-dimensional space. It should be noted that $(\rho, \boldsymbol{J})$ does not have to be the components of a rank-one Lorentz tensor for (3.2) to be valid.

We now apply (3.2) to the scalar field n-dimensional energy density, identifying ρ with

$$H = P^0 = \tfrac{1}{2}(\partial_0\psi^*\partial_0\psi + \partial_a\psi^*\partial_a\psi + m^2\psi^*\psi) \tag{3.3}$$

and J^a with

$$P^a = -\tfrac{1}{2}(\partial_0\psi^*\partial_a\psi + \partial_a\psi^*\partial_0\psi) = -\mathrm{Re}(\partial_0\psi^*\partial_a\psi) \tag{3.4}$$

Again it is known that the continuity equation

$$\partial_0 P^0 + \partial_a P^a = 0 \tag{3.5}$$

is satisfied assuming the n-dimensional Klein-Gordon equation.

If we define $H_{\text{in}}(t)$ to be the total energy inside $V(t)$, then the relationship (3.2) tells us that

$$\dot{H}_{\text{in}} = \int_{S(t)} [H(\boldsymbol{x}, t) - P_n(\boldsymbol{x}, t)]\, dS \tag{3.6}$$

where P_n is the normal component of $\boldsymbol{P}$.

We shall prove that $H \geqslant |P_n|\ \forall \boldsymbol{x}, t$.

Proof. (Note: we shall not use the repeated suffix convention in this proof.) From (3.3) and (3.4):

$$\begin{aligned} 4\left(H^2 - \sum_{a=1}^{n} (P^a)^2\right) &= |\partial_0\psi|^4 + \left(\sum_{a=1}^{n} |\partial_a\psi|^2\right)^2 - (\partial_0\psi^*)^2\left(\sum_{a=1}^{n} (\partial_a\psi)^2\right) \\ &\quad - \left(\sum_{a=1}^{n} (\partial_a\psi^*)^2\right)(\partial_0\psi)^2 + \text{non-negative terms} \\ &\geqslant \left(\sum_{a=1}^{n} |\partial_a\psi|^2\right)^2 - \left|\sum_{a=1}^{n} (\partial_a\psi)^2\right|^2 \qquad \text{by Schwarz's inequality} \\ &\geqslant 0 \qquad \text{by the triangle inequality.} \end{aligned}$$

So $H \geqslant |\boldsymbol{P}|$, hence $H \geqslant |P_n|$. It now follows from (3.6) that $\dot{H}_{\text{in}} \geqslant 0$; that is, the total energy outside $V(t)$ cannot increase.

4. Conclusion and discussion

The results of section 3 may be summarised by the following propositions and corollaries.

Proposition 1. Let ρ be a real-valued function defined on $\mathcal{M} \equiv \mathbb{R} \times \mathbb{R}^n$, the Minkowski space in some frame of reference, and $\boldsymbol{J} = (J^1, \ldots, J^n)$ be a real-valued space vector defined on $\mathcal{M}$. Suppose a finite region $V(t)$, surface $S(t)$, expands outwards with the speed of light, so that $S(t)$ is a 'light shell'. If

$$M(t) = \int_{V(t)} \rho(\boldsymbol{x}, t)\, \mathrm{d}V$$

and ρ, $\boldsymbol{J}$ satisfy the continuity equation (3.1), then the rate of change of M with respect to time is given by (3.2).

Corollary. If ρ is non-negative and $\rho \geq |\boldsymbol{J}|$ everywhere in $\mathcal{M}$ then $M(t)$ cannot decrease; equivalently, ρ integrated over the region outside the light shell cannot increase.

Proposition 2. The total energy outside the light shell of the scalar field governed by the Klein–Gordon equation cannot increase.

Corollary. The evolution of an isolated pulse of energy initially concentrated in a region $V(0)$ is causal; that is the energy remains concentrated in $V(t)$.

This last result is in contrast to the evolution of the Newton–Wigner probability density considered by Hegerfeldt and others, despite the fact that the evolution of both densities is governed by the Klein–Gordon equation. In the latter case the integrated probability outside the light shell may increase as is shown for a particular case by Rosenstein and Usher [5].

We can further relate these results to causality in the general case provided we are careful about the interpretation. Firstly, consider a classical relativistic fluid. The current $\boldsymbol{J}$ is $\rho\boldsymbol{v}$ where $\boldsymbol{v}$ is the velocity. The statement $\rho \geq |\boldsymbol{J}|$ is therefore equivalent to $|\boldsymbol{v}| \leq 1$ (that is, the velocity of the fluid is bounded by the velocity of light).

It is not possible to take this directly over to quantum mechanics, since it is not obvious that energy can be thought of as a fluid. For example, thinking naively of quanta as particles, we could have a few fast moving quanta whose presence is masked by many surrounding slow moving quanta, so that $|\boldsymbol{J}| \leq \rho$ is satisfied even if some quanta are superluminal. However, suppose we wish to transmit information, or produce an effect at a distance, by means of a pulse of energy whose flow is governed by the Klein–Gordon equation, and that in order to do this we need to be able to ignore the effect of the environment on the pulse. If this is the case we must assume that the motion of the energy pulse is essentially unaffected by the removal of all, or effectively all, the other energy. If the pulse is emitted from a region V at time $t = 0$, and $V(0) = V$, proposition 2 shows that the total energy outside the light shell remains small, so that no effect can be produced outside the light shell. In this sense causality is obtained.

A further point arises from the acausality theorems of Hegerfeldt and others. If the energy arises from a positive-energy solution of the Klein–Gordon equation, and the evolution is smooth, then the energy H takes the form (in the case $n = 1$)

$$H = \tfrac{1}{2}(|\hat{\omega}\psi|^2 + |\partial_x\psi|^2 + m^2|\psi|^2)$$

where $\hat{\omega} = F^{-1}\omega F$, F being the Fourier transformation operator, while $\omega F\psi(k) = (m^2 + k^2)^{1/2} F\psi(k)$. The energy can only be located in a bounded region V of space at any instant (in the given Lorentz frame) if $\hat{\omega}\psi$, $\partial_x\psi$ and ψ simultaneously vanish outside V. In the appendix we show this to be impossible. However, at time $t = 0$ the total energy will be small outside some sufficiently large region V. Proposition 2 then shows that the energy outside the light shell emanating from the surface of V at time $t = 0$ remains negligible.

Appendix

In this appendix we prove that $\hat{\omega}$ is non-local in the case $n = 1$. Our argument is essentially similar to that of Hegerfeldt [2], but we give it here for completeness.

Suppose $\phi = \hat{\omega}\psi$ where ψ has compact support. If ϕ also has compact support the Fourier transforms $F\phi$ and $F\psi$ are entire functions when extended to $\mathbb{C}$. Now by the definition of $\hat{\omega}$,

$$F\phi(z) = (m^2 + z^2)^{1/2} F\psi(z).$$

This contradicts the fact that

$$F\phi(z) = (z - \mathrm{i}m)^p \Phi(z) \qquad F\psi(z) = (z - \mathrm{i}m)^q \Psi(z)$$

where p and q are non-negative integers, while $\Phi(\mathrm{i}m)$ and $\Psi(\mathrm{i}m)$ are non-vanishing.

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